Stochastic recursive inclusion in two timescales with an application to the Lagrangian dual problem

Arun Selvan. R [1] and Shalabh Bhatnagar [2]

[1]arunselvan@csa.iisc.ernet.in
[2]shalabh@csa.iisc.ernet.in
[1,2]Department of Computer Science and Automation, Indian Institute of Science, Bangalore - 560012, India.

Abstract

In this paper we present a framework to analyze the asymptotic behavior of two timescale stochastic approximation algorithms including those with set-valued mean fields. This paper builds on the works of Borkar and Perkins & Leslie. The framework presented herein is more general as compared to the synchronous two timescale framework of Perkins & Leslie, however the assumptions involved are easily verifiable. As an application, we use this framework to analyze the two timescale stochastic approximation algorithm corresponding to the Lagrangian dual problem in optimization theory.

1 Introduction

The classical dynamical systems approach was developed by Benaïm [2, 3] and Benaïm and Hirsch [4]. They showed that the asymptotic behavior of a stochastic approximation algorithm (SA) can be studied by analyzing the asymptotics of the associated ordinary differential equation (*o.d.e.*). This method is popularly known as the *o.d.e. method* and was originally introduced by Ljung [12]. In 2005, Benaïm, Hofbauer and Sorin [5] extended the dynamical systems approach to include the situation where the stochastic approximation algorithm tracks a solution to the associated differential inclusion. Such algorithms are called *stochastic recursive inclusions*. For a detailed exposition on SA, the reader is referred to books by Borkar [8] and Kushner and Yin [11].

There are many applications where the aforementioned paradigms are inadequate. For example, the right hand side of a SA may require further averaging or an additional recursion to evaluate it. An instance mentioned in Borkar [7] is the 'adaptive heuristic critic' approach to reinforcement learning [10] that requires a stationary value iteration executed between two policy iterations. To solve such problems, Borkar [7] analyzed the two timescale SA algorithms. The

two timescale paradigm presented in Borkar [7] is inadequate if the coupled iterates are stochastic recursive inclusions. Such iterates arise naturally in many learning algorithms, see for instance *Section 5* of [13]. For another application from convex optimization the reader is referred to Section 4 of this paper. Such iterates also arise in applications that involve projections onto non-convex sets. The first attempt at tackling this problem was made by Perkins and Leslie [13] in 2012. They extended the two timescale scheme of Borkar [7] to include the situation when the two iterates track solutions to differential inclusions.

Consider the following coupled recursion:

$$\begin{aligned} x_{n+1} &= x_n + a(n)\left[u_n + M^1_{n+1}\right], \\ y_{n+1} &= y_n + b(n)\left[v_n + M^2_{n+1}\right], \end{aligned} \tag{1}$$

where $u_n \in h(x_n, y_n)$, $v_n \in g(x_n, y_n)$, $h : \mathbb{R}^{d+k} \to \left\{subsets\ of\ \mathbb{R}^d\right\}$ and $g : \mathbb{R}^{d+k} \to \left\{subsets\ of\ \mathbb{R}^k\right\}$. Such iterates were analyzed in [13]. Further, as an application a Markov decision process (MDP) based actor critic type learning algorithm was also presented in [13].

In this paper we generalize the synchronous two timescale stochastic approximation scheme presented in [13]. We present sufficient conditions that are mild and easily verifiable. For a complete list of assumptions used herein, the reader is referred to Section 2.2 and for the analyses under these conditions the reader is referred to Section 3. It is worth noting that the analysis of the faster timescale proceeds in a predictable manner, however, the *analysis of the slower timescale presented herein is new to the literature to the best of our knowledge.*

In convex optimization, one is interested in minimizing an objective function (that is convex) subject to a few constraints. A solution to this optimization problem is a set of vectors that minimize our objective function. Often this set is referred to as a minimum set. In Section 4, we analyze the two timescale SA algorithm corresponding to the Lagrangian dual of a primal problem. As we shall see later, this analysis considers a family of minimum sets and as a consequence of our framework these minimum sets are no longer required to be *singleton.* In [9], Dantzig, Folkman and Shapiro presented sufficient conditions for the continuity of minimum sets of continuous functions. We shall use results from that paper to show that under some standard convexity conditions the assumptions of Section 2.2 are satisfied. We then conclude from our main result, Theorem 3, that the two timescale algorithm in question converges to a solution to the dual problem.

2 Preliminaries and assumptions

2.1 Definitions and notations

The definitions and notations used in this paper are similar to those in Benaïm et. al. [5], Aubin et. al. [1] and Borkar [8]. We present a few for easy reference.

Let H be an upper semi-continuous, set-valued map on $\mathbb{R}^d$, where for any $x \in \mathbb{R}^d$, $H(x)$ is compact and convex valued. Note that we say that H is upper semi-continuous when $x_n \to x$, $y_n \to y$ and $y_n \in H(x_n)$ $\forall n$ implies $y \in H(x)$. Consider the differential inclusion (DI)

$$\dot{x} \in H(x). \tag{2}$$

We say that $\mathbf{x} \in \sum$ if $\mathbf{x}$ is an absolutely continuous map that satisfies (2). The *set-valued semiflow* Φ associated with (2) is defined on $[0, +\infty) \times \mathbb{R}^d$ as: $\Phi_t(x) = \{\mathbf{x}(t) \mid \mathbf{x} \in \sum, \mathbf{x}(0) = x\}$. Let $\mathcal{T} \times M \subset [0, +\infty) \times \mathbb{R}^k$ and define

$$\Phi_{\mathcal{T}}(M) = \bigcup_{t \in \mathcal{T},\ x \in M} \Phi_t(x).$$

$M \subseteq \mathbb{R}^d$ is *invariant* if for every $x \in M$ there exists a complete trajectory in M, say $\mathbf{x} \in \sum$ with $\mathbf{x}(0) = x$.

Let $x \in \mathbb{R}^d$ and $A \subseteq \mathbb{R}^d$, then $d(x, A) := \inf\{\|a - y\| \mid y \in A\}$. We define the *$\delta$-open neighborhood* of A by $N^\delta(A) := \{x \mid d(x, A) < \delta\}$. The *$\delta$-closed neighborhood* of A is defined by $\overline{N^\delta}(A) := \{x \mid d(x, A) \leq \delta\}$.

Let $M \subseteq \mathbb{R}^d$, the $\omega - limit\ set$ be given by $\omega_\Phi(M) := \bigcap_{t \geq 0} \overline{\Phi_{[t,+\infty)}(M)}$. Similarly the *limit set* of a solution $\mathbf{x}$ is given by $L(x) = \bigcap_{t \geq 0} \overline{\mathbf{x}([t, +\infty))}$.

$A \subseteq \mathbb{R}^d$ is an *attractor* if it is compact, invariant and there exists a neighborhood U such that for any $\epsilon > 0$, $\exists\ T(\epsilon) \geq 0$ such that $\Phi_{[T(\epsilon),+\infty)}(U) \subset N^\epsilon(A)$. Such a U is called the *fundamental neighborhood* of A. The *basin of attraction* of A is given by $B(A) = \{x \mid \omega_\Phi(x) \subset A\}$. If $B(A) = \mathbb{R}^d$, then the set is called a *globally attracting set*. It is called *Lyapunov stable* if for all $\delta > 0$, $\exists\ \epsilon > 0$ such that $\Phi_{[0,+\infty)}(N^\epsilon(A)) \subseteq N^\delta(A)$.

A set-valued map $h : \mathbb{R}^n \to \{subsets\ of\ \mathbb{R}^m\}$ is called a *Marchaud map* if it satisfies the following properties:

(i) For each $z \in \mathbb{R}^n$, $h(z)$ is convex and compact.

(ii) *(point-wise boundedness)* For each $z \in \mathbb{R}^n$, $\sup_{w \in h(z)} \|w\| < K(1 + \|z\|)$ for some $K > 0$.

(iii) h is an *upper semi-continuous* map.

The open ball of radius r around 0 is represented by $B_r(0)$, while the closed ball is represented by $\overline{B}_r(0)$.

2.2 Assumptions

Recall that we have the following coupled recursion:

$$x_{n+1} = x_n + a(n)\left[u_n + M^1_{n+1}\right],$$
$$y_{n+1} = y_n + b(n)\left[v_n + M^2_{n+1}\right],$$

where $u_n \in h(x_n, y_n)$, $v_n \in g(x_n, y_n)$, $h : \mathbb{R}^{d+k} \to \left\{subsets\ of\ \mathbb{R}^d\right\}$ and $g : \mathbb{R}^{d+k} \to \left\{subsets\ of\ \mathbb{R}^k\right\}$.

We list below our assumptions.

(A1) h and g are *Marchaud maps*.

(A2) $\{a(n)\}_{n\geq 0}$ and $\{b(n)\}_{n\geq 0}$ are two scalar sequences such that: $a(n), b(n) > 0$, for all n, $\sum\limits_{n\geq 0} (a(n) + b(n)) = \infty$, $\sum\limits_{n\geq 0} \left(a(n)^2 + b(n)^2\right) < \infty$ and $\lim_{n\to\infty} \frac{b(n)}{a(n)} = 0$. Without loss of generality, we let $\sup_n\ a(n),\ \sup_n\ b(n) \leq 1$.

(A3) $\{M^i_n\}_{n\geq 1}$, $i = 1, 2$, are square integrable martingale difference sequences with respect to the filtration $\mathcal{F}_n := \sigma\left(x_m, y_m, M^1_m, M^2_m :\ m \leq n\right)$, $n \geq 0$, such that $E[\|M^i_{n+1}\|^2 | \mathcal{F}_n] \leq K\left(1 + (\|x_n\| + \|y_n\|)^2\right)$, $i = 1, 2$, for some constant $K > 0$. Without loss of generality assume that the same constant, K, works for both $(A1)$ (in the property (ii) of Marchaud maps, see section 2.1) and $(A3)$.

(A4) $\sup_n \{\|x_n\| + \|y_n\|\} < \infty\ a.s.$

(A5) For each $y \in \mathbb{R}^k$, the differential inclusion $\dot{x}(t) \in h(x(t), y)$ has a globally attracting set, A_y, that is also Lyapunov stable. Further, $\sup\limits_{x\in A_y} \|x\| \leq K(1 + \|y\|)$. The set-valued map $\lambda : \mathbb{R}^k \to \{subsets\ of\ \mathbb{R}^d\}$, where $\lambda(y) = A_y$, is upper semi-continuous.

Define for each $y \in \mathbb{R}^k$, a function $G(y) := \overline{co}\left(\bigcup\limits_{x\in\lambda(y)} g(x, y)\right)$. The convex closure of a set $A \subseteq \mathbb{R}^k$, denoted by $\overline{co}(A)$, is closure of the convex hull of A, *i.e.*, the closure of the smallest convex set containing A. It will be shown later that G is a Marchaud map.

(A6) $\dot{y}(t) \in G(y(t))$ has a globally attracting set, A_0, that is also Lyapunov stable.

With respect to the faster timescale, the slower timescale iterates appear stationary, hence the faster timescale iterates track a solution to $\dot{x}(t) \in h(x(t), y_0)$, where y_0 is fixed (see Theorem 1). The y iterates track a solution to $\dot{y}(t) \in G(y(t))$ (see Theorem 2). It is worth noting that Theorems 1 & 2 only require $(A1) - (A5)$ to hold. Since $G(\cdot)$ is the convex closure of a union of compact convex sets one can expect the set-valued map to be point-wise bounded and convex. However, it is unclear why it should be upper semi-continuous (hence Marchaud). In lemma 2 we prove that G is indeed Marchaud without any additional assumptions.

Over the course of this paper we shall see that $(A5)$ is the key assumption that links the asymptotic behaviors of the faster and slower timescale iterates. It may be noted that $(A5)$ is *weaker* than the corresponding assumption - $(B6)/(B6)'$ used in [13]. For example, $(B6)'$ requires that $\lambda(y)$ and $\bigcup\limits_{x\in\lambda(y)} g(x, y)$ be *convex* for every $y \in \mathbb{R}^k$ while $(B6)$ requires that $\lambda(y)$ be singleton for every $y \in \mathbb{R}^k$. The reader is referred to [13] for more details. Note that $\lambda(y)$ being a singleton is a strong requirement in itself since it is the global attractor of some DI. It is observed in most applications that both $\lambda(y)$ and $\bigcup\limits_{x\in\lambda(y)} g(x, y)$ will not be convex and therefore $(B6)/(B6)'$ are easily violated. Further, our application discussed in Section 4 illustrates the same.

3 Proof of convergence

Before we start analyzing the coupled recursion given by (1), we prove a bunch of auxiliary results.

Lemma 1. *Consider the differential inclusion $\dot{x}(t) \in H(x(t))$, where $H : \mathbb{R}^n \to \mathbb{R}^n$ is a Marchaud map. Let A be the associated globally attracting set that is also Lyapunov stable. Then A is an attractor and every compact set containing A is a fundamental neighborhood.*

Proof. Since A is compact and invariant, it is left to prove the following: given a compact set $K \subseteq \mathbb{R}^n$ such that $A \subseteq K$; for each $\epsilon > 0$ there exists $T(\epsilon) > 0$ such that $\Phi_t(K) \subseteq N^\epsilon(A)$ for all $t \geq T(\epsilon)$.

Since A is Lyapunov stable, corresponding to $N^\epsilon(A)$ there exists $N^\delta(A)$, where $\delta > 0$, such that $\Phi_{[0,+\infty)}(N^\delta(A)) \subseteq N^\epsilon(A)$. Fix $x_0 \in K$. Since A is a globally attracting set, $\exists t(x_0) > 0$ such that $\Phi_{t(x_0)}(x_0) \subseteq N^{\delta/4}(A)$. Further, from the upper semi-continuity of flow it follows that $\Phi_{t(x_0)}(x) \subseteq N^{\delta/4}(\Phi_{t(x_0)}(x_0))$ for all $x \in N^{\delta(x_0)}(x_0)$, where $\delta(x_0) > 0$, see *Chapter 2* of Aubin and Cellina [1]. Hence we get $\Phi_{t(x_0)}(x) \subseteq N^\delta(A)$. Further since A is Lyapunov stable, we get $\Phi_{(t(x_0),+\infty]}(x) \subseteq N^\epsilon(A)$. In this manner for each $x \in K$ we calculate $t(x)$ and $\delta(x)$, the collection $\left\{N^{\delta(x)}(x) : x \in K\right\}$ is an open cover for K. Since K is compact, there exists a finite sub-cover $\left\{N^{\delta(x_i)}(x_i) \mid 1 \leq i \leq m\right\}$. For $T(\epsilon) := max\{t(x_i) \mid 1 \leq i \leq m\}$, we have $\Phi_{[T(\epsilon),+\infty)}(K) \subseteq N^\epsilon(A)$. $\square$

In Theorem 2 we prove that the slower timescale trajectory asymptotically tracks a solution to $\dot{y}(t) \in G(y(t))$. The following lemma ensures that the aforementioned DI has at least one solution.

Lemma 2. *The map G referred to in (A6) is a Marchaud map.*

Proof. Fix an arbitrary $y \in \mathbb{R}^k$. For any $x \in \lambda(y)$, it follows from $(A1)$ that

$$\sup_{z \in g(x,y)} \|z\| \leq K(1 + \|x\| + \|y\|).$$

From assumption $(A5)$, we have that $\|x\| \leq K(1 + \|y\|)$. Substituting in the above equation we may conclude the following:

$$\sup_{z \in g(x,y)} \|z\| \leq K\left(1 + K\left(1 + \|y\|\right) + \|y\|\right) = K(K+1)(1 + \|y\|) \ ,$$

$$\sup_{z \in \bigcup\limits_{x \in \lambda(y)} g(x,y)} \|z\| \leq K(K+1)(1 + \|y\|) \ ,$$

$$\sup_{z \in G(y)} \|z\| \leq K(K+1)(1 + \|y\|).$$

We have thus proven that G is point-wise bounded. From the definition of G, it follows that $G(y)$ is convex and compact.

It remains to show that G is an upper semi-continuous map. Let $z_n \to z$ and $y_n \to y$ in $\mathbb{R}^k$ with $z_n \in G(y_n)$, $\forall\ n \geq 1$. We need to show that $z \in G(y)$. We present a proof by contradiction. Since $G(y)$ is convex and compact, $z \notin G(y)$

implies that there exists a linear functional on $\mathbb{R}^k$, say f, such that $\sup\limits_{w \in G(y)} f(w) \leq \alpha - \epsilon$ and $f(z) \geq \alpha + \epsilon$, for some $\alpha \in \mathbb{R}$ and $\epsilon > 0$. Since $z_n \to z$, there exists N such that for all $n \geq N$, $f(z_n) \geq \alpha + \frac{\epsilon}{2}$. In other words, $G(y_n) \cap [f \geq \alpha + \frac{\epsilon}{2}] \neq \phi$ for all $n \geq N$. Here the notation $[f \geq a]$ is used to denote the set $\{x \mid f(x) \geq a\}$.

For the sake of convenience, we denote the set $\bigcup\limits_{x \in \lambda(y)} g(x, y)$ by $B(y)$. We claim that $B(y_n) \cap [f \geq \alpha + \frac{\epsilon}{2}] \neq \phi$ for all $n \geq N$. We prove this claim later, for now we assume that the claim is true and proceed. Pick $w_n \in g(x_n, y_n) \cap [f \geq \alpha + \frac{\epsilon}{2}]$, where $x_n \in \lambda(y_n)$ and $n \geq N$. It can be shown that $\{x_n\}_{n \geq N}$ and $\{w_n\}_{n \geq N}$ are norm bounded sequences and hence contain convergent subsequences. Construct sub-sequences, $\{w_{n(k)}\}_{k \geq 1} \subseteq \{w_n\}_{n \geq N}$ and $\{x_{n(k)}\}_{k \geq 1} \subseteq \{x_n\}_{n \geq N}$ such that $\lim\limits_{k \to \infty} w_{n(k)} = w$ and $\lim\limits_{k \to \infty} x_{n(k)} = x$. It follows from the upper semi-continuity of g that $w \in g(x, y)$ and from the upper semi-continuity of λ that $x \in \lambda(y)$, hence $w \in G(y)$. Since f is continuous, $f(w) \geq \alpha + \frac{\epsilon}{2}$. This is a contradiction.

It remains to prove that $B(y_n) \cap [f \geq \alpha + \frac{\epsilon}{2}] \neq \phi$ for all $n \geq N$. Suppose this were false, then $\exists \{m(k)\}_{k \geq 1} \subseteq \{n \geq N\}$ such that $B(y_{m(k)}) \subseteq [f < \alpha + \frac{\epsilon}{2}]$ for each $k \geq 1$. It can be shown that $\overline{co}(B(y_{m(k)})) \subseteq [f \leq \alpha + \frac{\epsilon}{2}]$ for each $k \geq 1$. Since $z_{m(k)} \to z$, $\exists N_1$ such that for all $m(k) \geq N_1$, $f(z_{m(k)}) \geq \alpha + \frac{3\epsilon}{4}$. This is a contradiction. Hence we get $B(x_n) \cap [f \geq \alpha + \frac{\epsilon}{2}] \neq \phi$ for all $n \geq N$. □

It is worth noting that $(A5)$ is a key requirement in the above proof. In the next lemma, we show the convergence of the martingale noise terms.

Lemma 3. *The sequences* $\{\zeta_n^1\}$ *and* $\{\zeta_n^2\}$, *where* $\zeta_n^1 = \sum_{m=0}^{n-1} a(m) M_{m+1}^1$ *and* $\zeta_n^2 = \sum_{m=0}^{n-1} b(m) M_{m+1}^2$, *are convergent almost surely.*

Proof. Although a proof of the above statement can be found in [2] or [8], we provide one for the sake of completeness. We only prove the almost sure convergence of ζ_n^1 as the convergence of ζ_n^2 can be similarly shown.

It is enough to show that

$$\sum_{m=0}^{\infty} a(m)^2 E\left[\|\zeta_{m+1}^1 - \zeta_m^1\|^2 | \mathcal{F}_m\right] \; < \; \infty \; a.s.,$$

$$i.e., \sum_{m=0}^{\infty} a(m)^2 E\left[\|M_{m+1}^1\|^2 | \mathcal{F}_m\right] \; < \; \infty \; a.s.$$

From assumption $(A3)$ it follows that

$$\sum_{m=0}^{\infty} a(m)^2 E\left[\|M_{m+1}^1\|^2 | \mathcal{F}_m\right] \; \leq \; K \sum_{m=0}^{\infty} a(m)^2 \left(1 + (\|x_m\| + \|y_m\|)^2\right).$$

From assumptions $(A2)$ and $(A4)$ it follows that

$$K \sum_{m=0}^{\infty} a(m)^2 \left(1 + (\|x_m\| + \|y_m\|)^2\right) < \infty \; a.s.$$

□

We now prove a couple of technical results that are essential to the proofs of Theorems 1 and 2.

Lemma 4. *Given any $y_0 \in \mathbb{R}^k$ and $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in N^{\delta}(\lambda(y_0))$, we have $g(x, y_0) \subseteq N^{\epsilon}(G(y_0))$.*

Proof. Assume the statement is not true. Then, $\exists\ \delta_n \downarrow 0$ and $x_n \in N^{\delta_n}(\lambda(y_0))$ such that $g(x_n, y_0) \nsubseteq N^{\epsilon}(G(y_0))$, $n \geq 1$. In other words, $\exists \gamma_n \in g(x_n, y_0)$ and $\gamma_n \notin N^{\epsilon}(G(y_0))$ for each $n \geq 1$. Since $\{x_n\}$ and $\{\gamma_n\}$ are bounded sequences there exist convergent sub-sequences, $\lim_{k\to\infty} x_{n(k)} = x$ and $\lim_{k\to\infty} \gamma_{n(k)} = \gamma$. Since $x_{n(k)} \in N^{\delta_{n(k)}}(\lambda(y_0))$ and $\delta_{n(k)} \downarrow 0$ it follows that $x \in \lambda(y_0)$ and hence $g(x, y_0) \subseteq G(y_0)$. We also have that $v \notin N^{\epsilon}(G(y_0))$ as $v_{n(k)} \notin N^{\epsilon}(G(y_0))$ for all $k \geq 1$. Since g is upper semi-continuous it follows that $\gamma \in g(x, y_0)$ and hence $\gamma \in G(y_0)$. This is a contradiction. □

Lemma 5. *Let $x_0 \in \mathbb{R}^d$ and $y_0 \in \mathbb{R}^k$ be such that the statement of lemma 4 is satisfied (with x_0 in place of x). If $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} y_n = y_0$ then $\exists N$ such that $\forall n \geq N$, $g(x_n, y_n) \subseteq N^{\epsilon}(G(y_0))$.*

Proof. If not, $\exists\ \{n(k)\} \subseteq \{n\}$ such that $\lim_{k\to\infty} n(k) = \infty$ and $g(x_{n(k)}, y_{n(k)}) \nsubseteq N^{\epsilon}(G(y_0))$. Without loss of generality assume that $\{n(k)\} = \{n\}$. In other words, $\exists \gamma_n \in g(x_n, y_n)$ such that $\gamma_n \notin N^{\epsilon}(G(y_0))$ for all $n \geq 1$. Since $\{\gamma_n\}$ is a bounded sequence, it has a convergent sub-sequence, *i.e.*, $\lim_{m\to\infty} \gamma_{n(m)} = \gamma$. Since $\lim_{m\to\infty} x_{n(m)} = x_0$, $\lim_{m\to\infty} y_{n(m)} = y_0$ and g is upper semi-continuous it follows that $\gamma \in g(x_0, y_0)$ and finally from lemma 4 we get that $\gamma \in N^{\epsilon}(G(y_0))$. This is a contradiction. □

Before we proceed let us construct trajectories, using (1), with respect to the faster timescale. Define $t(0) := 0$, $t(n) := \sum_{i=0}^{n-1} a(i)$, $n \geq 1$. The linearly interpolated trajectory $\overline{x}(t)$, $t \geq 0$, is constructed from the sequence $\{x_n\}$ as follows: let $\overline{x}(t(n)) := x_n$ and for $t \in (t(n), t(n+1))$, let

$$\overline{x}(t) := \left(\frac{t(n+1) - t}{t(n+1) - t(n)}\right) \overline{x}(t(n)) + \left(\frac{t - t(n)}{t(n+1) - t(n)}\right) \overline{x}(t(n+1)). \quad (3)$$

We construct a piecewise constant trajectory from the sequence $\{u_n\}$ as follows: $\overline{u}(t) := u_n$ for $t \in [t(n), t(n+1))$, $n \geq 0$.

Let us construct trajectories with respect to the slower timescale in a similar manner. Define $s(0) := 0$, $s(n) := \sum_{i=0}^{n-1} b(i)$, $n \geq 1$. Let $\widetilde{y}(s(n)) := y_n$ and for $s \in (s(n), s(n+1))$, let

$$\widetilde{y}(s) := \left(\frac{s(n+1) - s}{s(n+1) - s(n)}\right) \widetilde{y}(s(n)) + \left(\frac{s - s(n)}{s(n+1) - s(n)}\right) \widetilde{y}(s(n+1)). \quad (4)$$

Also $\widetilde{v}(s) := v_n$ for $s \in [s(n), s(n+1))$, $n \geq 0$, is the corresponding piecewise constant trajectory.

For $s \geq 0$, let $x^s(t)$, $t \geq 0$, denote the solution to $\dot{x}^s(t) = \overline{u}(s+t)$ with the initial condition $x^s(0) = \overline{x}(s)$. Similarly, let $y^s(t)$, $t \geq 0$, denote the solution to $\dot{y}^s(t) = \widetilde{v}(s+t)$ with the initial condition $y^s(0) = \widetilde{y}(s)$.

The y iterate in recursion (1) can be re-written as

$$y_{n+1} = y_n + a(n) \left[\frac{b(n)}{a(n)} v_n + \frac{b(n)}{a(n)} M^2_{n+1} \right]. \tag{5}$$

Define $\epsilon(n) := \frac{b(n)}{a(n)} v_n$ and $M^3_{n+1} = \frac{b(n)}{a(n)} M^2_{n+1}$. It can be shown that the stochastic iteration given by $y_{n+1} = y_n + a(n)M^3_{n+1}$ satisfies the set of assumptions given in Benaïm [2]. From $(A1)$, $(A2)$ and $(A4)$ it follows that $\epsilon(n) \to 0$ almost surely. Since $\epsilon(n) \to 0$ the recursion given by (5) and $y_{n+1} = y_n + a(n)M^3_{n+1}$ have the same asymptotics. For a precise statement and proof the reader is referred to lemma 2.1 of [7].

Define $\overline{y}(t(n)) := y_n$, where $n \geq 0$ and $\overline{y}(t)$ for $t \in (t(n), t(n+1))$ by

$$\overline{y}(t) := \left(\frac{t(n+1) - t}{t(n+1) - t(n)} \right) \overline{y}(t(n)) + \left(\frac{t - t(n)}{t(n+1) - t(n)} \right) \overline{y}(t(n+1)). \tag{6}$$

The trajectory $\overline{y}(\cdot)$ can be seen as an evolution of the y iterate with respect to the faster timescale, $\{a(n)\}$.

Lemma 6. *Almost surely every limit point, $y(\cdot)$, of $\{\overline{y}(s+\cdot) \mid s \geq 0\}$ in $C([0,\infty), \mathbb{R}^k)$ as $s \to \infty$ satisfies $y(t) = y(0)$, $t \geq 0$.*

Proof. It can be shown that $y_{n+1} = y_n + a(n)M^3_{n+1}$ satisfies the assumptions of Benaïm [2]. Hence the corresponding linearly interpolated trajectory tracks the solution to $\dot{y}(t) = 0$. The statement of the lemma then follows trivially. □

Lemma 7. *For any $T > 0$, $\lim\limits_{s\to\infty} \sup\limits_{t \in [0,T]} \|\overline{x}(s+t) - x^s(t)\| = 0$ and $\lim\limits_{s\to\infty} \sup\limits_{t\in[0,T]} \|\widetilde{y}(s+t) - y^s(t)\| = 0$, a.s.*

Proof. In order to prove the above lemma, it enough to prove the following:

$$\lim_{t(n)\to\infty} \sup_{0 \leq t(n+m) - t(n) \leq T} \|\overline{x}(t(n+m)) - x^{t(n)}(t(n+m) - t(n))\| = 0 \text{ and}$$

$$\lim_{s(n)\to\infty} \sup_{0 \leq s(n+m) - s(n) \leq T} \|\widetilde{y}(s(n+m)) - y^{s(n)}(s(n+m) - s(n))\| = 0 \; a.s.$$

Note the following:

$$\overline{x}(t(n+m)) = \overline{x}(t(n)) + \sum_{k=0}^{m-1} \left[a(n+k) \left(\overline{u}(t(n+k)) + M^1_{n+k+1} \right) \right],$$

$$x^{t(n)}(t(n+m) - t(n)) = \overline{x}(t(n)) + \int_0^{t(n+m)-t(n)} \overline{u}(t(n)+z) \; dz,$$

$$x^{t(n)}(t(n+m) - t(n)) = \overline{x}(t(n)) + \int_{t(n)}^{t(n+m)} \overline{u}(z) \; dz. \tag{7}$$

From (7), we get,

$$\|\overline{x}(t(n+m)) - x^{t(n)} \left(t(n+m) - t(n) \right)\| =$$

$$\left\| \sum_{k=0}^{m-1} a(n+k)\overline{u}(t(n+k)) - \sum_{k=0}^{m-1} \int_{t(n+k)}^{t(n+k+1)} \overline{u}(z) \; dz + \sum_{k=0}^{m-1} a(n+k) M^1_{n+k+1} \right\|.$$

The $R.H.S.$ of the above equation equals $\left\|\sum_{k=0}^{m-1} a(n+k)M^1_{n+k+1}\right\|$ as

$$\sum_{k=0}^{m-1} a(n+k)\overline{u}(t(n+k)) \ = \sum_{k=0}^{m-1} \int_{t(n+k)}^{t(n+k+1)} \overline{u}(z)\ dz.$$

Since $\zeta^1_n := \sum_{m=0}^{n-1} a(m)M^1_{m+1},\ n \geq 1$, converges $a.s.$, the first part of claim follows.

The second part, for the y iterates, can be similarly proven. $\square$

From assumptions $(A1)$ and $(A4)$ it follows that $\{x^r(\cdot) \mid r \geq 0\}$ and $\{y^r(\cdot) \mid r \geq 0\}$ are equicontinuous and pointwise bounded families of functions. By the Arzela-Ascoli theorem they are relatively compact in $C([0,\infty),\mathbb{R}^d)$ and $C([0,\infty),\mathbb{R}^k)$ respectively. From lemma 7 it then follows that $\{\overline{x}(r+\cdot) \mid r \geq 0\}$ and $\{\widetilde{y}(r+\cdot) \mid r \geq 0\}$ are also relatively compact, see (3) and (4) for the definitions of $\overline{x}(\cdot)$ and $\widetilde{y}(\cdot)$, respectively.

3.1 Convergence in the faster timescale

The following theorem and its proof are similar to Theorem 2 from Chapter 5 of Borkar [8]. We present a proof for the sake of completeness.

Theorem 1. *Almost surely, every limit point of* $\{\overline{x}(r+\cdot) \mid r \geq 0\}$ *in* $C([0,\infty),\mathbb{R}^d)$ *is of the form* $x(t) = x(0) + \int_0^t u(z)\ dz$*, where* u *is a measurable function such that* $u(t) \in h(x(t), y(0))$*,* $t \geq 0$*, for some fixed* $y(0) \in \mathbb{R}^k$*.*

Proof. Fix $T > 0$, then $\{\overline{u}(r+t) \mid t \in [0,T]\}$, $r \geq 0$ can be viewed as a subset of $L_2([0,T],\mathbb{R}^d)$. From $(A1)$ and $(A4)$ it follows that the above is uniformly bounded and hence weakly relatively compact. Let $\{r(n)\}$ be a sequence such that the following hold:

(i) $\lim_{n\to\infty} r(n) = \infty$.

(ii) There exists some $x(\cdot) \in C([0,\infty),\mathbb{R}^d)$ such that $\overline{x}(r(n)+\cdot) \to x(\cdot)$ in $C([0,\infty),\mathbb{R}^d)$. This is because $\{\overline{x}(r+\cdot) \mid r \geq 0\}$ is relatively compact in $C([0,\infty),\mathbb{R}^d)$.

(iii) $\overline{y}(r(n)+\cdot) \to y(\cdot)$ in $C([0,\infty),\mathbb{R}^k)$ for some $y \in C([0,\infty),\mathbb{R}^k)$. It follows from lemma 6 that $y(t) = y(0)$ for all $t \geq 0$.

(iv) $\overline{u}(r(n)+\cdot) \to u(\cdot)$ weakly in $L_2([0,T],\mathbb{R}^d)$.

From lemma 7, it follows that $x^{r(n)}(\cdot) \to x(\cdot)$ in $C([0,\infty),\mathbb{R}^d)$, and we have that $\int_0^t \overline{u}(r(n)+z)\ dz\ \to \int_0^t u(z)\ dz$ for $t \in [0,T]$. Letting $n \to \infty$ in

$$x^{r(n)}(t) = x^{r(n)}(0) + \int_0^t \overline{u}(r(n)+z)\ dz,\ \ t \in [0,T],$$

we get $x(t) = x(0) + \int_0^t u(z)\ dz,\ \ t \in [0,T]$.

Since $\overline{u}(r(n)+\cdot) \to u(\cdot)$ weakly in $L_2([0,T],\mathbb{R}^d)$, there exists $\{n(k)\} \subset \{n\}$ such that $n(k) \uparrow \infty$ and

$$\frac{1}{N}\sum_{k=1}^{N} \overline{u}(r(n(k))+\cdot) \to u(\cdot)$$

strongly in $L_2([0,T],\mathbb{R}^d)$. Further, there exist $\{N(m)\} \subset \{N\}$ such that $N(m) \uparrow \infty$ and

$$\frac{1}{N(m)} \sum_{k=1}^{N(m)} \overline{u}(r(n(k))+\cdot\,) \to u(\cdot\,) \tag{8}$$

a.e. in $[0,T]$.

Define $[t] := max\{t(n) \mid t(n) \le t\}$. If we fix $t_0 \in [0,T]$ such that (8) holds, then $\overline{u}(r(n(k)) + t_0) \in h(\overline{x}([r(n(k)) + t_0]), \overline{y}([r(n(k)) + t_0]))$ for $k \ge 1$. Since $\lim_{n(k)\to\infty} \|\overline{x}(r(n(k))+t_0) - \overline{x}([r(n(k))+t_0])\| = 0$, it follows that $\lim_{k\to\infty} \overline{x}([r(n(k)) + t_0]) = x(t_0)$, and similarly, we have that $\lim_{k\to\infty} \overline{y}([r(n(k)) + t_0]) = y(0)$. Since h is upper semi-continuous it follows that $\lim_{k\to\infty} d\left(\overline{u}(r(n(k)) + t_0), h(x(t_0), y(0))\right) = 0$. The set $h(x(t_0), y(0))$ is compact and convex, hence it follows from (8) that $u(t_0) \in h(x(t_0), y(0))$. ☐

3.2 Convergence in the slower timescale

Theorem 2. *For any $\epsilon > 0$, almost surely any limit point of $\{\widetilde{y}(r+\cdot\,) \mid r \ge 0\}$ in $C([0,\infty),\mathbb{R}^k)$ is of the form $y(t) = y(0) + \int_0^t v(z)\, dz$, where v is a measurable function such that $v(t) \in N^\epsilon(G(y(t)))$, $t \ge 0$.*

Proof. Fix $T > 0$. As before let $\{r(n)\}_{n\ge1}$ be a sequence such that the following hold:

(i) $\lim_{n\to\infty} r(n) = \infty$.

(ii) $\widetilde{y}(r(n)+\cdot\,) \to y(\cdot\,)$ in $C([0,\infty),\mathbb{R}^k)$, where $y(\cdot\,) \in C([0,\infty),\mathbb{R}^k)$.

(iii) $\widetilde{v}(r(n)+\cdot\,) \to v(\cdot\,)$ weakly in $L_2([0,T],\mathbb{R}^k)$.

Also, as before, we have the following:

(i) There exists $\{n(k)\} \subseteq \{n\}$ such that $\frac{1}{N}\sum_{k=1}^{N} \widetilde{v}(r(n(k))+\cdot\,) \to v(\cdot\,)$ strongly in $L_2([0,T],\mathbb{R}^d)$ as $N \to \infty$.

(ii) There exist $\{N(m)\} \subset \{N\}$ such that $N(m) \uparrow \infty$ and

$$\frac{1}{N(m)} \sum_{k=1}^{N(m)} \widetilde{v}(r(n(k))+\cdot\,) \to v(\cdot\,) \tag{9}$$

a.e. on $[0,T]$.

Define $[s]' := max\{s(n) \mid s(n) \le s\}$. Construct a sequence $\{m(n)\}_{n\ge1} \subseteq \mathbb{N}$ such that $s(m(n)) = [r(n) + t_0]'$ for each $n \ge 1$. Observe that $\overline{y}(t(m(n))) = \widetilde{y}(s(m(n)))$ and $\widetilde{v}(r(n) + t_0) \in g(\overline{x}(t(m(n))), \overline{y}(t(m(n))))$.

Choose $t_0 \in (0,T)$ such that (9) is satisfied. If we show that $\exists\, N$ such that for all $n \ge N$, $g(\overline{x}(t(m(n))), \overline{y}(t(m(n)))) \subseteq N^\epsilon(G(y(t_0)))$ then (9) implies that $v(t_0) \in \overline{N^\epsilon}(G(y(t_0)))$.

It remains to show the existence of such a N. We present a proof by contradiction. We may assume without loss of generality that for each $n \ge 1$, $g(\overline{x}(t(m(n))), \overline{y}(t(m(n)))) \nsubseteq N^\epsilon(G(y(t_0)))$, *i.e.*, $\exists\, \gamma_n \in g(\overline{x}(t(m(n))), \overline{y}(t(m(n))))$ such that $\gamma_n \notin N^\epsilon(G(y(t_0)))$. Let S_1 be the set on which $(A4)$ is satisfied

and S_2 be the set on which lemma 3 holds. Clearly $P(S_1 \cap S_2) = 1$. For each $\omega \in S_1 \cap S_2$, $\exists\ R(\omega) < \infty$ such that $\sup_n \|x_n(\omega) + y_n(\omega)\| \leq R(\omega)$ and $\sup_n\ K(1 + \|y_n(\omega)\|) \leq R(\omega)$. In what follows we merely use R and the dependence on ω (sample path) is understood to be implicit. From lemma 1 it follows that corresponding to $\dot{x}(t) \in h(x(t), y(t_0))$ and some $\delta > 0$ there exists T_0, possibly dependent on R, such that for all $t \geq T_0$, $\Phi_t(x_0) \in N^{\delta}(\lambda(y(t_0)))$ for all $x_0 \in \overline{B}_R(0)$.

We construct a new sequence $\{l(n)\}_{n\geq 1}$ from $\{m(n)\}_{n\geq 1}$ such that $t(l(n)) = min\{t(m) \mid |t(m(n)) - t(m)| \leq T_0\}$. Since $\{\overline{x}(r + \cdot\,) \mid r \geq 0\}$ is relatively compact in $C([0, \infty), \mathbb{R}^d)$, it follows that $\overline{x}(t(l(n)) + \cdot\,) \to x(\cdot\,)$ in $C([0, T_0], \mathbb{R}^d)$. From lemma 6 we can conclude that $\overline{y}(t(l(n)) + \cdot\,) \to y(\cdot\,)$ in $C([0, T_0], \mathbb{R}^k)$, where $y(t) = y(t_0)$ for all $t \in [0, T_0]$. lemma 6 only asserts that the limiting function is a constant, we recognize this constant to be $y(t_0)$ since $\|\overline{y}(t(l(n)) + T_0) - \overline{y}(t(m(n)))\| \to 0$ and $\overline{y}(t(l(n)) + T_0) \to y(t_0)$. Note that in the foregoing discussion we can only assert the existence of convergent subsequences, again for the sake of convenience we assume that the sequences at hand are both convergent. It follows from Theorem 1 that $x(t) = x(0) + \int_0^t u(z)\ dz$, where $u(t) \in h(x(t), y(t_0))$. Since $x(0) \in \overline{B}_R(0)$ it follows that $x(T_0) \in N^{\delta}(\lambda(y(t_0)))$.

From lemma 4 we get $g(x(T_0), y(t_0)) \subseteq N^{\epsilon}(G(y(t_0)))$. Since $\|\overline{x}(t(m(n))) - \overline{x}(t(l(n)) + T_0)\| \to 0$ it follows that $\overline{x}(t(m(n))) \to x(T_0)$. It follows from lemma 5 that $\exists N$ such that for $n \geq N$, $g(\overline{x}(t(m(n))), \overline{y}(t(m(n)))) \subseteq N^{\epsilon}(G(y(t_0)))$. This is a contradiction. $\square$

A direct consequence of the above theorem is that almost surely any limit point of $\{\widetilde{y}(r + \cdot\,) \mid r \geq 0\}$ in $C([0, \infty), \mathbb{R}^k)$ is of the form $y(t) = y(0) + \int_0^t v(z)\ dz$, where v is a measurable function such that $v(t) \in G(y(t))$, $t \geq 0$.

3.3 Main result

Theorem 3. *Under assumptions $(A1) - (A6)$, almost surely the set of accumulation points is given by*

$$\left\{(x, y) \mid \varliminf_{n\to\infty} d\left((x, y), (x_n, y_n)\right) = 0\right\} \subseteq \bigcup_{y \in A_0} \left\{(x, y) \mid x \in \lambda(y)\right\}. \tag{10}$$

Proof. The statement follows directly from Theorems 1 and 2. $\square$

Note that assumption $(A6)$ allows us to narrow the set of interest. If $(A6)$ does not hold then we can only conclude that the $R.H.S.$ of (10) is $\bigcup_{y \in \mathbb{R}^k} \{(x, y) \mid x \in \lambda(y)\}$. On the other hand if $(A6)$ holds and A_0 consists of a single point, say y_0, then the $R.H.S.$ of (10) is $\{(x, y_0) \mid x \in \lambda(y_0)\}$. Further, if $\lambda(y_0)$ is of cardinality one then the $R.H.S.$ of (10) is just $(\lambda(y_0), y_0)$.

Remark: It may be noted that all proofs and conclusions in this paper will go through if $(A1)$ is weakened to let g be upper semi-continuous and $g(x, \cdot\,)$ be Marchaud on $\mathbb{R}^k$ for each fixed $x \in \mathbb{R}^d$.

4 Application: An SA algorithm to solve the Lagrangian dual problem

Let $f : \mathbb{R}^d \to \mathbb{R}$ and $g : \mathbb{R}^d \to \mathbb{R}^k$ be two given functions. We want to minimize $f(x)$ subject to the condition that $g(x) \leq 0$ (every component of $g(x)$ is non-positive). This problem can be stated in the following primal form:

$$\underset{x \in \mathbb{R}^d}{inf} \; \underset{\substack{\mu \in \mathbb{R}^k \\ \mu \geq 0}}{sup} \left(f(x) + \mu^T g(x) \right). \tag{11}$$

Let us consider the following two timescale SA algorithm to solve the primal (11):

$$\begin{aligned} \mu_{n+1} &= \mu_n + a(n) \left[\nabla_\mu \left(f(x_n) + \mu_n^T g(x_n) \right) + M^1_{n+1} \right], \\ x_{n+1} &= x_n - b(n) \left[\nabla_x \left(f(x_n) + \mu_n^T g(x_n) \right) + M^2_{n+1} \right]. \end{aligned} \tag{12}$$

where, $a(n), b(n) > 0$, $\sum_{n \geq 0} a(n) = \sum_{n \geq 0} b(n) = \infty$, $\sum_{n \geq 0} a(n)^2 < \infty$, $\sum_{n \geq 0} b(n)^2 < \infty$ and $\frac{b(n)}{a(n)} \to 0$. Without loss of generality assume that $\sup\limits_n a(n),\ b(n) \leq 1$. The sequences $\{M^1_n\}_{n \geq 1}$ and $\{M^2_n\}_{n \geq 1}$ are suitable martingale difference noise terms.

Suppose there exists $x_0 \in \mathbb{R}^d$ such that $g(x_0) \geq 0$, then $\mu = (\infty, \ldots, \infty)$ maximizes $f(x_0) + \mu^T g(x_0)$. With respect to the faster timescale (μ) iterates the slower timescale (x) iterates can be viewed as being "quasi-static", see [8] for more details. It then follows from the aforementioned observation that the μ iterates cannot be guaranteed to be stable. In other words, we cannot use (12) to solve the primal problem.

If strong duality holds then solving (11) is equivalent to solving its dual given by:

$$\underset{\substack{\mu \in \mathbb{R}^k \\ \mu \geq 0}}{sup} \; \underset{x \in \mathbb{R}^d}{inf} \left(f(x) + \mu^T g(x) \right). \tag{13}$$

Further, the two timescale scheme to solve the dual problem is given by:

$$\begin{aligned} x_{n+1} &= x_n - a(n) \left[\nabla_x \left(f(x_n) + \mu_n^T g(x_n) \right) + M^2_{n+1} \right], \\ \mu_{n+1} &= \mu_n + b(n) \left[\nabla_\mu \left(f(x_n) + \mu_n^T g(x_n) \right) + M^1_{n+1} \right]. \end{aligned} \tag{14}$$

Note that (14) is obtained by flipping the timescales of (12). Strong duality can be enforced if we assume the following:

(S1) $f(x) = x^T Q x + b^T x + c$, where Q is a positive semi-definite $d \times d$ matrix, $b \in \mathbb{R}^d$ and $c \in \mathbb{R}$.

(S2) $g = A$, where A is a $k \times d$ matrix.

(S3) f is bounded from below.

The reader is referred to Bertsekas [6] for further details. For the purposes of this section we assume the following:

$(S1)-(S3)$ are satisfied.

$(A3)'$ $\sum_{n \geq 0} a(n) M^i_{n+1} < \infty$ a.s., where $i = 1,\ 2$.

The sole purpose of $(A3)$ in Section 2.2 is to ensure the convergence of the martingale noise terms *i.e.*, $(A3)'$ holds. It is clear that (14) satisfies $(A1)$ since $(S1)-(S3)$ hold while $(A2)$ is the step size assumption that is enforced.

The stability of the μ iterates in (14) directly follows from strong duality and $(A3)'$. The μ iterates are "quasi-static" with respect to the x iterates. Further, since $f(x) + \mu_0^T g(x)$ is a convex function (from $(S1)$ and $(S2)$), for a fixed μ_0, $f(x) + \mu_0^T g(x)$ achieves its minimum "inside" $\mathbb{R}^d$. Hence, the stability of the x iterates will follow from that of the μ iterates and $(A3)'$. In other words, (14) satisfies $(A1)$, $(A2)$, $(A3)'$ & $(A4)$, see Section 2.2 for the definitions of $(A1), (A2)$ and $(A4)$.

For a fixed μ_0, the minimizers of $f(x) + \mu_0^T g(x)$ constitute the global attractor of the o.d.e., $\dot{x}(t) = -\nabla_x (f(x) + \mu_0^T g(x))$. Our paradigm comes in handy when this attractor set is *NOT singleton*, which is generally the case. In other words, we can define the following set valued map: $\lambda_m : \mathbb{R}^k \to \mathbb{R}^d$, where $\lambda_m(\mu_0)$ is the global attractor of $\dot{x}(t) = -\nabla_x (f(x) + \mu_0^T g(x))$.

Now we check that (14) satisfies $(A5)$. To do so it is enough to ensure that λ_m is an upper semi-continuous map. Recall that $\lambda_m(\mu)$ is the minimum set of $f(x) + \mu^T g(x)$ for each $\mu \in \mathbb{R}^k$. Dantzig, Folkman and Shapiro [9] studied the continuity of minimum sets of continuous functions. A wealth of sufficient conditions can be found in [9] which when satisfied by the functions guarantee "continuity" of the corresponding minimum sets. In our case since $(S1)-(S3)$ are satisfied, *Corollary I.2.3* of [9] guarantees upper semi-continuity of λ_m.

Since (A1)-(A5) are satisfied by (14), it follows from Theorems 1 & 2 that:

(I) Almost surely every limit point of $\{\overline{x}(r + \cdot\,) \mid r \geq 0\}$ in $C([0,\infty), \mathbb{R}^d)$ is of the form $x(t) = x(0) + \int_0^t \nabla_x (f(x(t)) + \mu_0^T g(x(t)))\, dt$ for some $x(0) \in \mathbb{R}^d$ and some $\mu_0 \in \mathbb{R}^k$.

(II) Almost surely, any limit point of $\{\widetilde{\mu}(r + \cdot\,) \mid r \geq 0\}$ in $C([0,\infty), \mathbb{R}^k)$ is of the form $\mu(t) = \mu(0) + \int_0^t \nu(z)\, dz$ for some measurable function ν with $\nu(t) \in G(\mu(t))$, $t \geq 0$ and $G(\mu(t)) = \overline{co}\left(\{\nabla_\mu (f(x) + \mu(t)^T g(x)) \mid x \in \lambda_m(\mu(t))\}\right)$.

For the construction of $\overline{x}(\cdot\,)$ and $\widetilde{\mu}(\cdot\,)$ see equations (3) and (4) respectively. If in addition, (14) satisfies $(A6)$ *i.e.,* $\exists\ A_\mu \subset \mathbb{R}^k$ such that it is the global attractor of $\mu(t) \in G(\mu(t))$, then it follows from Theorem 3 that: almost surely any accumulation point of $\{(x_n, y_n) \mid n \geq 0\}$ belongs to the set $\mathcal{A} := \underset{\mu \in A_m}{\cup} \{(x, \mu) \mid x \in \lambda_m(\mu)\}$. The attractor A_μ is the maximum set of $H(\mu) := \underset{x \in \mathbb{R}^d}{inf} \left(f(x) + \mu^T g(x)\right)$ subject to $\mu \geq 0$. It may be noted that H is a concave function that is bounded above as a consequence of strong duality. For any $(x^*, \mu^*) \in \mathcal{A}$ we have that

$$f(x^*) + (\mu^*)^T g(x^*) = \underset{\substack{\mu \in \mathbb{R}^k \\ \mu \geq 0}}{sup}\ \underset{x \in \mathbb{R}^d}{inf} f(x) + \mu^T g(x).$$

In other words, almost surely the two timescale iterates given by (14) converge to a solution of the dual (13). It follows from strong duality that they almost surely converge to a solution of the primal (11).

5 Conclusions

In this paper we have presented a framework for the analysis of two timescale stochastic approximation algorithms with set valued mean fields. Our framework generalizes the one by Perkins and Leslie. We note that the analysis of the faster timescale proceeds in a predictable manner but the analysis of the slower timescale is new to the literature to the best of our knowledge. As an application we analyze the two timescale scheme that arises from the Lagrangian dual problem in optimization using our framework. Our framework is applicable even when the minimum sets are not singleton.

References

[1] J. Aubin and A. Cellina. *Differential Inclusions: Set-Valued Maps and Viability Theory*. Springer, 1984.

[2] M. Benaïm. A dynamical system approach to stochastic approximations. *SIAM J. Control Optim.*, 34(2):437–472, 1996.

[3] M. Benaïm. Dynamics of stochastic approximation algorithms. *Séminaire de Probabilités XXXIII, Lecture Notes in Mathematics, Springer*, 1709:1–68, 1999.

[4] M. Benaïm and M. W. Hirsch. Asymptotic pseudotrajectories and chain recurrent flows, with applications. *J. Dynam. Differential Equations*, 8:141–176, 1996.

[5] M. Benaïm, J. Hofbauer, and S. Sorin. Stochastic approximations and differential inclusions. *SIAM Journal on Control and Optimization*, pages 328–348, 2005.

[6] D.P. Bertsekas. *Convex Optimization Theory*. Athena Scientific optimization and computation series. Athena Scientific, 2009.

[7] V. S. Borkar. Stochastic approximation with two time scales. *Syst. Control Lett.*, 29(5):291–294, 1997.

[8] V. S. Borkar. *Stochastic Approximation: A Dynamical Systems Viewpoint*. Cambridge University Press, 2008.

[9] G. B. Dantzig, J. Folkman, and N. Shapiro. On the continuity of the minimum set of a continuous function. *Journal of Mathematical Analysis and Applications*, 17(3):519 – 548, 1967.

[10] Sathiya S. Keerthi and B. Ravindran. A tutorial survey of reinforcement learning. *Indian Academy of Sciences, Proc. in Engineering Sciences*, 19:851–889, 1994.

[11] H. Kushner and G.G. Yin. *Stochastic Approximation and Recursive Algorithms and Applications*. Springer, 2003.

[12] L. Ljung. Analysis of recursive stochastic algorithms. *Automatic Control, IEEE Transactions on*, 22(4):551–575, 1977.

[13] S. Perkins and D.S. Leslie. Asynchronous stochastic approximation with differential inclusions. *Stochastic Systems*, 2(2):409–446, 2012.